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MA204 - Linear Algebra and Matrices
Problem Sheet - 3

Vector Spaces and Subspaces
and
Solving $Ax = b$ and $Ax = 0$

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1. Which of the following subsets of R^3 are actually subspaces?
   (a) The plane of vectors (b_1, b_2, b_3) with first component $b_1 = 0$.
   (b) The plane of vectors b with $b_1 = 1$.
   (c) The vectors b with $b_2 b_3 = 0$ (this is the union of two subspaces, the plane $b_2 = 0$ and the plane $b_3 = 0$).
   (d) All combinations of two given vectors $(1, 1, 0)$ and $(2, 0, 1)$.
   (e) The plane of vectors (b_1, b_2, b_3) that satisfy $b_3 - b_2 + 3b_1 = 0$.

2. Which of the following are subspaces of R^∞?
   (a) All sequences like $(1, 0, 1, 0, \ldots)$ that include infinitely many zeros.
   (b) All sequences $(x_1, x_2, \ldots)$ with $x_j = 0$ from some point onward.
   (c) All decreasing sequences: $x_{j+1} \leq x_j$ for each j.
   (d) All convergent sequences: the x_j have a limit as $j \to \infty$.
   (e) All arithmetic progressions: $x_{j+1} - x_j$ is the same for all j.
   (f) All geometric progressions $(x_1, kx_1, k^2x_1, \ldots)$ allowing all k and x_1.

3. Let P be the plane in 3-space with equation $x + 2y + z = 6$. What is the equation of the plane P_0 through the origin parallel to P? Are P and P_0 subspaces of R^3?

4. The matrix $A = \begin{bmatrix} 2 & -2 \\ 2 & -2 \end{bmatrix}$ is a "vector" in the space M of all 2 by 2 matrices. Write the zero vector in this space, the vector $\frac{1}{2}A$, and the vector $-A$. What matrices are in the smallest subspace containing A?

5. Find the value of c that makes it possible to solve $Ax = b$, and solve it:
$$\begin{aligned} u + v + 2w &= 2 \\ 2u + 3v - w &= 5 \\ 3u + 4v + w &= c. \end{aligned}$$

6. Construct a system with more unknowns than equations, but no solution. Change the right-hand side to zero and find all solutions x_n.

7. Find the complete solutions of

$$\begin{aligned} x+3y+3z&=1\\ 2x+6y+9z&=5\\ -x-3y+3z&=5 \end{aligned} \quad \text{and} \quad \begin{bmatrix}1&3&1&2\\2&6&4&8\\0&0&2&4\end{bmatrix}\begin{bmatrix}x\\y\\z\\t\end{bmatrix}=\begin{bmatrix}1\\3\\1\end{bmatrix}.$$

8. Give examples of matrices A for which the number of solutions to $Ax = b$ is

   (a) 0 or 1, depending on b.

   (b) ∞, regardless of b.

   (c) 0 or ∞, depending on b.

   (d) 1, regardless of b.

9. Show by example that these three statements are generally *false*:

   (a) A and A^T have the same nullspace.

   (b) A and A^T have the same free variables.

   (c) If R is the reduced form $rref(A)$ then R^T is $rref(A^T)$.

10. Construct a matrix whose column space contains $(1, 1, 5)$ and $(0, 3, 1)$ and whose nullspace contains $(1, 1, 2)$.

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